The Base Measure Problem and its Solution

Example: What is the density after stretching a distribution on the circle?

\[ x, y \sim \text{uniform}_\text{on_unit_circle} \]
\[ x', y' = 2x, 20y \]
\[ p(x', y') = ? \]

Easy, right?
- \[ p(x, y) = 1/2\pi \text{ when } x^2 + y^2 = 1 \]
- Let \( f(x, y) = (2x, 20y) \)
- \[ |\det J_f| = 40 \text{ everywhere} \]
- Ergo \( p(x', y') = 1/80\pi \text{ when } (x'/2)^2 + (y'/20)^2 = 1 \)

Wrong!
Perimeter of ellipse \( (x'/2)^2 + (y'/20)^2 = 1 \) is about 81.28, much less than \( 80\pi \).

The distribution isn't uniform!

Right answer

The density \( 1/2\pi \) is with respect to Lebesgue measure on the circle, not all of \( \mathbb{R}^2 \).
- Circle's tangent at \( (x, y) \) is \((-y, x)\).
- Directional derivative of \( f \) is \((-2y, 20x)\).
- Change of arclength is \( \sqrt{4y^2 + 400x^2} \).
- \( p(x', y') = 1/2\pi\sqrt{100x'^2 + y'^2}/100 \).

In general:

\[ p(x') = p(x) \sqrt{\det(VV^T)/\det(V'V'^T)} \]
where \( v_i \) is an arbitrary basis for the tangent space, and \( v'_i \) are those directional derivatives of \( f \)

When does this happen?
Whenever the base measure matters and is not Lebesgue on \( \mathbb{R}^n \).
- Transforming discrete distributions embedded in \( \mathbb{R}^n \).
- Transforming distributions on symmetric matrices, simplexes, spheres, etc.
- Reversible-jump MCMC on any of the above.
- MCMC or SMC with discrete + continuous observation model (e.g., Indian GPA problem).

Computation:
- Log Jacobian determinants of bijections not enough.
- Explicitly represent tangent space of support.
- Automatic differentiation to compute directional derivatives.
- Two-argument dispatch or Visitor pattern to cover efficient special cases.